Stars Summary Notes

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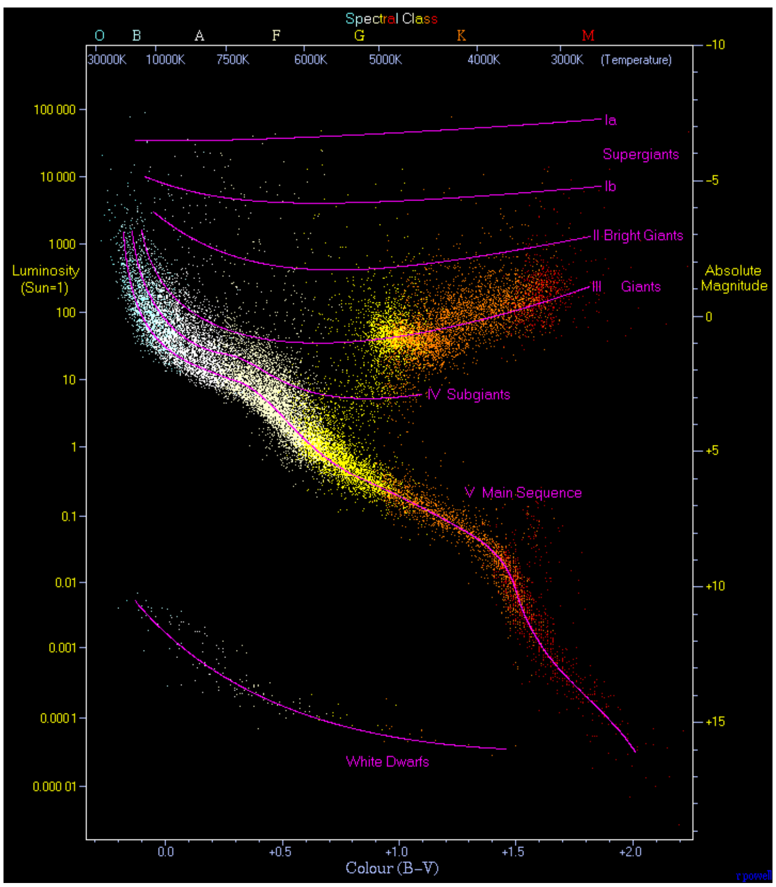


Contents

1 Some Observed Properties

Apparent magnitude given by:

$$m_1 - m_2 = -2.5 \log_{10}(f_1/f_2), \tag{1}$$

absolute magnitude given by:

$$m - M = 5 \log_{10}(d_{\mathrm{pc}}) - 5, \tag{2}$$

luminosity is related to flux by

$$L = 4\pi d^2 f, \tag{3}$$

where d is the distance from the observer to the emitter.

A body in thermodynamic equilibrium emits as a black-body. Stars are black-body emitters. Wien's displacement law states that the black-body radiation curve for different temperatures will peak at different wavelengths that are inversely proportional to the temperature,

$$\lambda_{\mathrm{max}} = \frac{b}{T}, \tag{4}$$

where $b = 2.9 \times 10^{-3}$ is Wein's displacement constant.

Note that stars are not perfect black-body emitters, we therefore define an effective temperature, T_{e}, in such a way that a black-body with the same radius as the star at this temperature would radiate the same amount of energy.

The Stefan-Boltzmann law states that the radiation emitted by a black-body per square metre of surface, F, is given by

$$F = \sigma T^4, \tag{5}$$

where σ is the Stefan-Boltzmann constant.

Radii of stars can be measured via eclipses, or via inteferemotry.

Binary stars are classified depending on how they are identified:

- Visual Binary - both stars can be spatially resolved
- Astrometric Binary - infer from motion of brighter star
- Eclipsing Binary - one star may pass infront of the other, reducing its observed brightness for a period of time
- Spectroscopic Binary - periodic shift in the spectral lines observed

We can determine masses using the generalised form of Kepler's 3rd law:

$$P^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)}, \tag{6}$$

where a is the sum of the semi-major axes of each elliptical orbit.

If we know the ratio of the semi-major axes of the ellipses, we can work out the individual masses, as

$$\frac{m_1}{m_2} = \frac{a_1}{a_2}. \tag{7}$$

Note that we can only measure the angular separation of visual binaries, we need the distance to the system to derive the semi-major axes.

From spectroscopy, we can measure the radial velocities of the stars in terms of their inclination angle.

$$v_{\text{max}} = v \sin i \tag{8}$$

where i is the angle of inclination.

For main-sequence stars, we have a tight relationship between mass and luminosity,

$$\frac{L}{L_\odot} \approx \left(\frac{M}{M_\odot}\right)^a, \tag{9}$$

where a is approximately 3.5.

2 Spectral properties, excitation, ionisation

To understand the spectral properties of stars, we consider both excitation and ionisation.

2.1 Excitation

The number of electrons in each orbital is calculated assuming the Maxwell-Boltzmann velocity distribution (i.e., assuming that gas is in thermodynamic equilibrium).

The ratio of probability between energy states s_a and s_b is given by the Boltzmann factor,

$$\frac{P(s_b)}{P(s_a)} = e^{-(E_b - E_a)/kT}. \tag{10}$$

Orbitals can be populated with more than one electron, so we account for this degeneracy using statistical weights,

$$\frac{N_b}{N_a} = \frac{g_b}{g_a} e^{-(E_b - E_a)/kT}. \tag{11}$$

2.2 Ionisation

Ionisation can occur when the absorbed energy exceeds the energy required to eject the electron from the atom.

The Saha equation relates the ionization state of a gas in thermal equilibrium to its temperature and pressure,

$$\frac{N_{i+1}}{N_i} = \frac{2Z_{i+1}}{n_e Z_i} \left(\frac{2\pi m_e kT}{h^2} \right)^{\frac{3}{2}} e^{-\chi_i/kT}, \tag{12}$$

where Z_i is the partition function for energy level i, n_e is the number of free electrons per unit volume, and χ_i is the energy required to remove an electron from an atom in state i.

3 Hydrostatic Equilibrium and Conditions in the Stellar Core

Stars do not typically noticeably contract or expand, they are luminous, and they live for a long time, therefore they require:

- Balance between gravity and pressure to provide stability
- Continuous luminous energy source to provide long-lasting emission

3.1 Hydrostatic Equilibrium

Hydrostatic equilibrium is balance between gravity and pressure,

$$\frac{\partial^2 r}{\partial t^2} = \frac{-Gm}{r^2} - 4\pi r^2 \frac{\partial P}{\partial m}. \tag{13}$$

Applying conservation of mass to the mass in and flowing through a shell in the star, we derive the 1st equation of stellar structure,

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho. \tag{14}$$

Considering the forces on such a shell, we derive the 2nd equation of stellar structure,

$$\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{-Gm\rho}{r^2}. \tag{15}$$

3.2 Pressure at the Stellar Core

Using the hydrostatic equation (Eq. 13), and assuming constant density, we can estimate the stellar-core pressure,

$$P_c \approx \frac{3}{8\pi} \frac{GM^2}{R^4}. \tag{16}$$

3.3 Gaseous state of stellar material

Note that the high temperatures and pressures in stars mean that stars are made out of plasma, i.e., a gas whose behaviour is dominated by the electromagnetic interaction between the charged particles.

We define the mean molecular mass as the average mass of a free particle in the gas divided by the mass of a hydrogen atom,

$$\mu = \frac{\overline{m}}{m_H}. \tag{17}$$

More generally, define μ based on the fraction of Hydrogen, X, Helium, Y, and metals, Z,

$$\frac{1}{\mu} = 2X + \frac{3}{4}Y + \frac{1}{2}Z. \tag{18}$$

Considering the pressure inside a star as the sum of gas and radiation pressures, we can write

$$P = nkT + \frac{1}{3}aT^4, \tag{19}$$

or alternatively,

$$P = \frac{\rho kT}{\mu m_H} + \frac{1}{3}aT^4, \tag{20}$$

where $a = 7.57 \times 10^{-16}\mathrm{Jm^{-3}K^{-4}}$ is the radiation density constant.

The first term gives the gas pressure and follows from the ideal gas law, the second gives the radiation pressure which follows from the photon flux.

4 Virial Theorem and Gravitational Collapse

4.1 The Virial Theorem

The Virial theorem describes the overall connection between the gravitational potential energy and kinetic energy of a system.

Starting with (Eq. 13), we multiply it by the volume then integrate by parts to get the generalised form of the Virial theorem, the Gravitational potential is given by

$$U = -3 \int_0^M \frac{P}{\rho} \mathrm{d}m. \tag{21}$$

Assuming the ideal gas law, we find the kinetic energy per unit mass in the star is given by

$$E_{\mathrm{KE}} = \frac{3P}{2\rho}, \tag{22}$$

substituting into (Eq. 21) and integrating, we get the internal energy of the star,

$$K = -\frac{1}{2}U. \tag{23}$$

The Virial theorem implies that as the star collapses under gravity, only half of the potential energy is radiated away.

Note that the Virial theorem applies generally to systems in equilibrium, and has general formula

$$K = -\frac{1}{3}\phi U, \tag{24}$$

where ϕ has non-relativistic and relativistic values of 3/2 and 3, respectively.

4.2 Gravitational Collapse

We find the total energy available for gravitational collapse by integrating the gravitational potential over the whole star,

$$U_g \approx -\frac{3}{5}\frac{GM^2}{R}. \tag{25}$$

By the Virial theorem, only half of this energy is radiates, the rest goes into heating the star.

4.3 Timescales of Stellar Evolution

The dynamical timescale is the approximate time it would take for a star to collapse if its pressure suddenly ceased,

$$\tau_{\mathrm{dyn}} \approx \sqrt{\frac{R^3}{GM}}. \tag{26}$$

The thermal (Kelvin-Helmholtz) timescale is the approximate time it would take a star to radiate away its total kinetic energy at its current luminosity rate,

$$\tau_{\rm KH} \approx \frac{GM^2}{2LR}. \tag{27}$$

The nuclear timescale is the time it will take a star to use up all its nuclear fuel,

$$\tau_{\rm nuc} \approx \frac{E_{\rm NUC}}{L}. \tag{28}$$

5 Equations of State

5.1 Local Thermal Equilibrium

The mean free path of a photon is given by

$$l_{\mathrm{ph}} = \frac{1}{\kappa\rho}, \tag{29}$$

the difference in temperature a photon observes at the start and the end of one mean free path, in the Sun, is

$$\Delta T = \frac{\mathrm{d}T}{\mathrm{d}r} l_{\mathrm{ph}} \approx 10^{-4}, \tag{30}$$

so on small scales, stars are in thermal equilibrium.

5.2 Equations of State

For an ideal gas, pressure is given by

$$P = nkT = \frac{k}{\mu m_u}\rho T. \tag{31}$$

In extremely dense objects, electrons are forced to be extremely close to each other. The Pauli exclusion principle and Heisenberg uncertainty principle cause electron degeneracy pressure. In non-relativistic conditions, this is given by

$$P_e = K_{\mathrm{NR}} \left(\frac{\rho}{\mu_e}\right)^{\frac{5}{3}}, \tag{32}$$

and, in extremely relativistic conditions, by

$$P_e = K_{\mathrm{ER}} \left(\frac{\rho}{\mu_e}\right)^{\frac{4}{3}}. \tag{33}$$

Radiation pressure is given by

$$P_{\mathrm{rad}} = \frac{1}{3}aT^4, \tag{34}$$

as above.

The pressure at the centre of a white dwarf can be calculated from

$$P \approx \frac{1}{3}n_e p v, \tag{35}$$

where n_e is electron number density, p is momentum (assumed to be the same for all particles), and v is velocity. Rewriting and substituting expressions for these values, we find

$$P \approx \frac{\hbar^2}{m_e}\left[\frac{Z\rho}{Am_H}\right]^{\frac{5}{3}} \tag{36}$$

5.3 Lane-Emden Equation

The Lane-Emden equation is a dimensionless form of Poisson's equation for the gravitational field of a spherically symmetric, Newtonian self-gravitating polytropic fluid.

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)+\theta^n=0, \tag{37}$$

where ξ is a dimensionless radius, θ is related to the density by $\rho=\rho_c\theta^n$, and n is the polytropic index,

$$P=K\rho^{1+1/n}. \tag{38}$$

To solve the Lane-Emden equation, we assume the star is supported by both ideal gas and radiation pressure, such that

$$\frac{P_g}{P_g+P_r}=\beta, \tag{39}$$

substituting in, we find

$$P=\left(\frac{k_B}{\mu m_H}\right)^{\frac{4}{3}}\left(\frac{3(1-\beta)}{\alpha\beta^4}\right)^{\frac{1}{3}}\rho^{\frac{4}{3}}, \tag{40}$$

this is a polytropic equation of state.

6 Nuclear Fusion

6.1 Binding Energy

Total mass of a nucleus is less than the mass of its constituent nucleons. The mass lost results in a release of 'binding' energy. The binding energy is required to break the nucleus into its constituent parts - it gives atoms stability.

For a nucleus consisting of Z protons, and N neutrons, its binding energy is given by

$$E_b(Z, N) = \Delta m c^2 = \left[Z m_p + N M_n - m(Z, N)\right] c^2. \tag{41}$$

Binding energy is the basis behind nuclear fusion. Only elements lighter than Iron can undergo nuclear fusion, heavier elements undergo fission instead. This means that Iron is the heaviest element that can be made from fusion in stars, all heavier elements are made in supernovae.

6.2 Temperature Required for Fusion

The Coulomb barrier gives the repulsive potential between two nuclei with proton number Z_1, Z_2 respectively. We can equate this to mean particle kinetic energy,

$$\frac{3}{2} k_B T_{\text{classical}} = \frac{1}{4\pi\varepsilon_0} \frac{Z_1 Z_2 e_c^2}{r}. \tag{42}$$

Rearranging gives

$$T_{\text{classical}} = \frac{Z_1 Z_2 e_c^2}{6\pi\varepsilon_0 k_B r} \tag{43}$$

Setting r as the typical radius of a nucleus, we get $T_{\text{classical}} \approx 10^{10}$K for Hydrogen-Hydrogen fusion. This is much higher than the core temperature of the Sun, so fusion is classically impossible - we need to consider quantum effects to explain it.

If, instead, we assume the particles are behaving as waves with de Broglie wavelength $\lambda = \frac{h}{p}$, we can rewrite the kinetic energy equation in terms of momentum and wavelength,

$$\frac{1}{2}\mu_m v^2 = \frac{p^2}{2\mu_m} = \frac{h^2}{\lambda^2 2\mu_m} = \frac{1}{4\pi\varepsilon_0} \frac{Z_1 Z_2 e_c^2}{\lambda}. \tag{44}$$

Rearranging for λ, and subbing in λ for r into (Eq. 43), we find

$$T_{\text{quantum}} = \frac{Z_1^2 Z_2^2 e_c^4 \mu_m}{12\pi^2 \varepsilon_0^2 k_B h^2} \approx 10^7\text{k}. \tag{45}$$

This temperature is consistent with the estimated temperature for the solar core, this is approximately the minimum temperature required for nuclear fusion. This process is referred to as quantum tunneling.

6.3 Probability of Fusion Occurring

The probability of nuclear reaction occurring is a product of two factors,

- The probability of two particles getting close enough for the nuclear force to become important
- The probability that a nuclear reaction will then occur

The first factor depends on the masses and charges of the particles, the number of particles present, and the temperature. The nuclei must overcome Coulomb barriers, and get close enough to have a chance of interacting.

The second factor depends on the properties of the nuclei involved. The nuclear fusion probability is the product of the tunneling probability and the particle velocity probability (from the Maxwell-Boltzmann distribution).

The Gamow peak is the peak of the Probability-Energy distribution for fusion, and occurs at the energy at which the probability of reaction is highest.

6.4 Nuclear Reaction Rates

We can characterise the rate of nuclear reactions in the form of a power law centred at a particular temperature. For a two-particle reaction, particles i and x,

$$r_{ix} \approx r_0 X_i X_x \rho^{\alpha'} T^{\beta}, \tag{46}$$

where X gives the mass fraction of a particle, r_0 is constant, α' and β are to be determined, $\alpha' = 2$ for a two body collision. Combining this with the energy released per reaction, we determine the energy released per second as

$$\varepsilon_{ix} \approx \varepsilon_0' X_i X_x \rho^{\alpha} T^{\beta}, \tag{47}$$

where $\alpha = \alpha' - 1$. The main factors in nuclear fusion are therefore density and temperature.

7 Nucleosynthesis and Neutrinos

7.1 Conservation Laws

Three quantities are conserved in nuclear reactions:

1. Charge
2. Nucleon number - number of protons and neutrons
3. Lepton number - electrons and neutrinos contribute +1, positrons and anti-neutrinos contribute -1.

7.2 Proton-Proton (PP) Chain

The proton-proton (PP) chain is the dominant nuclear fusion process in stars like the Sun, it converts Hydrogen into Helium. There are three possible paths in the PP-chain, PPI, II and III.

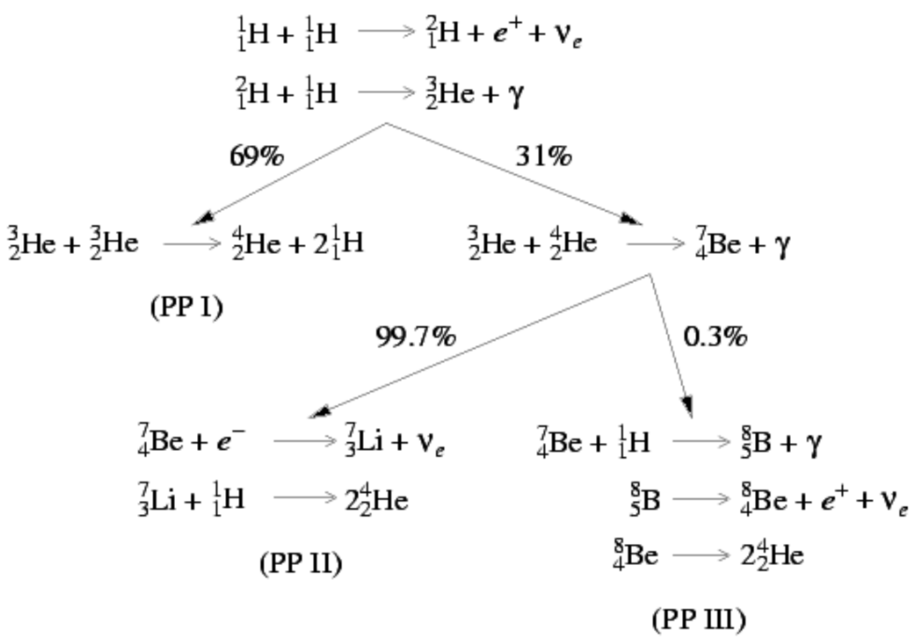


Figure 1: The three possible paths in the PP chain, the probability of each path is shown.

The first reaction shown is the basis for the whole PP chain. It resembles the β-decay of the neutron. A proton 'decays' near another proton to form a deuteron. The second reaction is so fast that its rate is unimportant. The third reaction in the PPI chain completes the PP chain in about 69% of PP chain terminations, the total energy release is 26.7MeV.

The reaction between ^{3}He and ^{4}He leads to some very important neutrino producing interactions. The majority of ^{7}Be produced in this way is then burnt via the PPII chain, only very little is burnt via the PPIII chain.

The rate of energy release varies smoothly with temperature, over a limited range in temperature, one can write

$$\varepsilon_{\rm pp} = \varepsilon_1 X_H^2 \rho T^4. \tag{48}$$

7.3 Carbon-Nitrogen-Oxygen (CNO) Cycle

This is another Helium producing chain, but with CNO as catalysts.

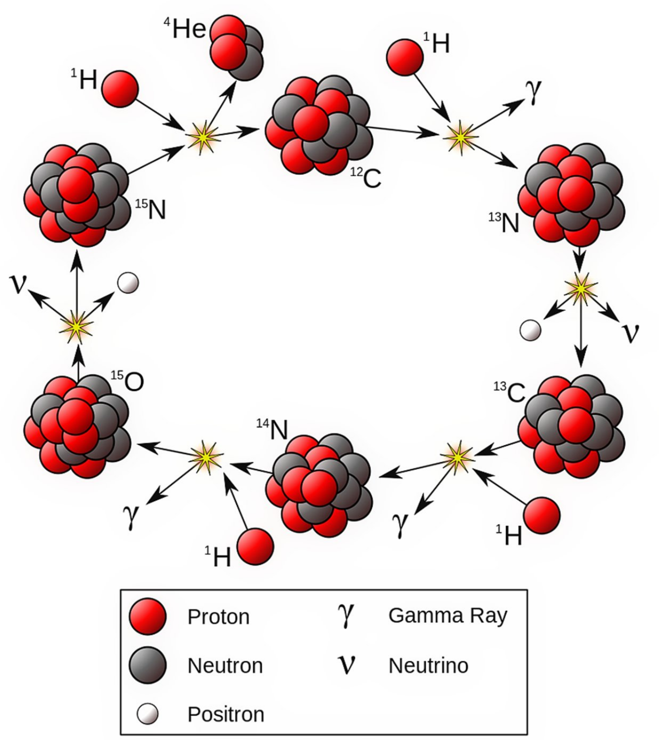


Figure 2: The CNO cycle

It contributes only 1.5% of the total solar luminosity, but in higher temperature stars ($M > 1.2M_\odot$), it is more important. The total energy release is the same as for the PP-chain.

The rate of energy release is similar to that of the PP-chain,

$$\varepsilon_{\rm CNO} = \varepsilon_2 X_H X_C \rho T^{17}. \tag{49}$$

Over the lifetime of a star, heavier elements are produced via fusion, meaning the Coulomb barrier becomes significantly higher, requiring higher core temperatures. The most massive stars will ultimately reach very high temperatures, $> 10^9$K.

7.4 Triple Alpha Process

Not important for main sequence stars, but becomes important at higher temperatures. This is the fusion of 2 ^{4}He nuclei to form ^{8}Be, then the fusion of that with another ^{4}He nucleus to produce ^{1}2C.

The rate of energy generation is given by

$$\varepsilon_{3\alpha} = \varepsilon_3 X_{He}^3 \rho^2 T^{40}. \tag{50}$$

Heavier elements are produced in different nuclear burning cycles at higher temperatures.

7.5 Testing Nuclear Fusion at the Stellar Core

We cannot test nuclear fusion at the stellar core via radiation, as photon transport takes 10^6 years, and they change in energy and wavelength. We instead use neutrinos.

The cross sectional area of a typical neutrino is $\sigma \approx 10^{-48}$m^2, meaning it is able to escape stars very quickly (2 seconds to escape the Sun). Measuring the energy of neutrinos from different parts of a star allows us to confirm our theories of nuclear fusion in stars.

8 Energy Transport

8.1 The Production and Conservation of Energy

Can define energy release $\mathrm{d}L$ as

$$\mathrm{d}L = \varepsilon \mathrm{d}m, \tag{51}$$

where ε is energy released per unit mass by nuclear reactions as given in previous lectures.

For a spherically symmetric star, the mass of a thin shell of depth $\mathrm{d}r$ is

$$\mathrm{d}m = \rho \mathrm{d}V = 4\pi r^2 \rho \mathrm{d}r, \tag{52}$$

and therefore,

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \rho \varepsilon. \tag{53}$$

This is the equation of energy conservation, is positive through regions of energy generation (stellar core) and is 0 outside of the core.

Opacity drives the transfer of photon energy. The main methods of energy transport in stars are:

1. Radiation - energy transport by emission and absorption of photons
2. Convection - energy transport by mass motion of gas
3. Conduction - only important in dense, degenerate stars

8.2 Radiation

Photons make a random walk through the sun with mean free path l given by

$$l = \frac{vt}{n\sigma vt} = \frac{1}{n\sigma}, \tag{54}$$

where σ is the effective cross sectional area for collision.

The number N of randomly directed steps is given by

$$N = \left(\frac{d}{l}\right)^2, \tag{55}$$

where d is the net displacement after N steps.

8.3 Convection

We can express the equation for radiation pressure (Eq. 34) as

$$\frac{\mathrm{d}P_{\mathrm{rad}}}{\mathrm{d}r} = -\frac{\kappa\rho F_{\mathrm{rad}}}{c}, \tag{56}$$

where κ is the opacity. Combining these two equations, we find

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{4ac}\frac{\kappa\rho F_{\mathrm{rad}}}{T^3} = -\frac{3}{16\pi ac}\frac{\kappa\rho L_{\mathrm{rad}}}{T^3 r^2}. \tag{57}$$

In the core, where temperature is highest, radiation dominates, but further out the temperature gradient is greater, as shown by the T^{-3} term, and convection dominates.

9 Opacity

Opacity is the resistance by material to the flow of radiation, it depends on chemical composition, density, temperature, and, importantly, photon wavelength.

If I_λ is the photon intensity at wavelength λ, then

$$\mathrm{d}I_\lambda = -\kappa_\lambda \rho I_\lambda \mathrm{d}s. \tag{58}$$

Integrating through the column density of gas, in the case of uniform gas density, we find the observed intensity to be given by

$$I_\lambda = I_{\lambda,0} e^{-\kappa_\lambda \rho s} = I_{\lambda,0} e^{-\tau_\lambda}, \tag{59}$$

where τ_λ is the optical depth.

The main sources of opacity are:

1. Bound-Bound transitions
2. Bound-Free transitions
3. Free-Free emission
4. Electron scattering

$$\kappa = \kappa_{\mathrm{bb}} + \kappa_{\mathrm{bf}} + \kappa_{\mathrm{ff}} + \kappa_{\mathrm{es}} \tag{60}$$

The Rosseland mean opacity is obtained by averaging opacity over all frequencies for convenience, so we don't have to track individual photons.

$$\kappa = \kappa_0 \rho^\alpha T^\beta \tag{61}$$

9.1 Bound-Bound Transitions

Occur when a photon is absorbed and an electron transitions into a higher energy level. This is most common at lower temperatures, as photons must have energy corresponding to electron orbital energies, so is most common in stellar atmospheres and less common in stellar cores, where the atoms are ionised.

Bound-Bound transitions produce line opacity, as only photons at energy corresponding to transitions are absorbed.

If the electron de-excites down to the original level, a photon of the original energy is emitted in a random direction, however often electrons cascade down different orbital energies, emitting many photons at lower energies - degradation of photon energies.

There is no simple equation for bound-bound transitions that describes all contributions to the opacity.

9.2 Bound-Free Transitions

Photons with energy greater than the ionisation energy can free electrons, excess photon energy contributes to the electron kinetic energy.

Most common at moderate temperatures, where a large fraction of the atoms are only partially ionised (around $10^4 < T < 10^6$K.

The reverse process is recombination, an electron is captured and photons are emitted as it cascades down through orbital energies.

This is an example of continuum opacity, as photons can be absorbed at a range of energies.

$$\kappa_{\mathrm{bf}} = \kappa_0 \rho T^{-3.5} \tag{62}$$

9.3 Free-Free Transitions

This is a result of the interactions between a free electron, a photon, and an ion. It causes the electron to gain velocity at the expense of the photon, the ionised atom is required to conserve energy and momentum.

Another example of continuum opacity, this is most effective at moderate temperatures in stellar interiors where atoms are partially ionised, since there is a high availability of electrons close to ions.

The reverse is Bremsstrahlung emission, the electron is decelerated and emits a photon with wavelength equal to the electron energy loss.

$$\kappa_{\mathrm{ff}} = \kappa_0 \rho T^{-3.5}, \tag{63}$$

note κ_0 different for bound-free and free-free.

9.4 Electron Scattering

The photon is scattered due to an interaction with an electron, the scattering is based on the Thomson cross section of an electron.

$$\sigma_T = \frac{1}{6\pi\varepsilon_0^2}\left(\frac{e_c^2}{m_e c^2}\right)^2 \approx 6.65 \times 10^{-29}\mathrm{m}^2. \tag{64}$$

Electron scattering is important at high temperatures ($T > 10^6$K), where other forms of opacity are minimally effective, i.e., when atoms are completely ionised and there is a high electron density; no temperature dependence. This is another form of continuum opacity.

Compton scattering occurs at even higher temperatures, $T > 10^8$K, when an electron loosely bound to an atom gains energy at the expense of the photon.

$$\kappa_{\mathrm{es}} = \frac{\sigma_T n_e}{\rho} \tag{65}$$

Note, each part of the star is dominated by one of these sources of opacity, but since photons spend the most time in dense sections where electron scattering is most significant, opacity due to electron scattering dominates.

10 Convection

Radiation and conduction occur when there is a temperature gradient. Convection only occurs in liquids and gases when the temperature gradient exceeds a critical value. Convection is thought to be a very efficient method of energy transport and chemical mixing in stars.

10.1 Schwarzschild Criterion for Convection

Consider a mass element $\mathrm{d}m$ rising out of the star by a small amount. Since the pressure in a star decreases with increasing radius $P_2 < P_1$, the surrounding gas pressure will be lower than the pressure of the mass element.

The mass element will therefore expand until it reaches equilibrium with its surroundings, given difference in dynamical and thermal timescales, no head exchange occurs during this process, it is adiabatic expansion. Therefore, the gas pressure in the mass element is

$$P = K_a \rho^{\gamma}, \tag{66}$$

K_a constant, γ the ratio of specific heats. Mass element will rise if it is less dense than its surroundings - unstable against convection, or sink if denser - stable against convection.

The Schwarzchild criterion gives that for convection to dominate,

$$\left|\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{\mathrm{sur}} > \left(\frac{\gamma - 1}{\gamma}\right) \frac{T}{P} \left|\frac{\mathrm{d}P}{\mathrm{d}r}\right|_{\mathrm{sur}} \tag{67}$$

must be satisfied.

Recall the formula for temperature gradient (Eq. 57), we therefore expect to see convection in:

1. Regions with high opacity, such as regions close to the stellar surface in less-massive stars
2. Regions where sufficient energy is produced to cause strong increase in luminosity, such as in the cores of luminous and massive stars, where the CNO cycle dominates (recall T^{17} dependence on energy output).

In the Sun's core, we can show that energy generation is too low to cause convection.

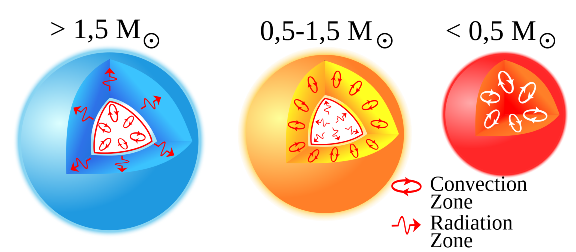


Figure 3: Diagram shows convection and radiation zones in different mass stars on the main sequence.

10.2 Mixing Length

The mixing length theory of convection is the leading model of convection in stars.

A hot rising bubble is expected to travel a distance before it will thermalise with the surroundings, giving up its excess heat at a constant pressure. This distance is called the mixing length, and is defined as

$$l = \alpha H_P, \tag{68}$$

where H_P is the scale height,

$$\frac{1}{H_P} = -\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}r}, \tag{69}$$

and therefore

$$P = P_0 e^{-r/H_P}. \tag{70}$$

The scale height is the distance over which pressure declines by a factor of e.

Numerical models typically find $0.5 \lesssim \alpha \lesssim 3$, and we often assume $\alpha \approx 1$.

Recalling that, in hydrostatic equilibrium, the pressure gradient for a star is given by (Eq. 15),

$$H_P = \frac{r^2 P}{\rho G M_r}, \tag{71}$$

so

$$l = \alpha \frac{r^2 P}{\rho G M_r} \approx \frac{r^2 P}{\rho G M_r}. \tag{72}$$

For the Sun, we estimate the mixing length for convection cells near the surface to be ($0.03 R_\odot \lesssim l \lesssim 0.17 R_\odot$).

11 Mass Limits

The minimum mass of a star is set by the requirement for fusion to occur, so $T > 10^7$K and $M > 0.1M_\odot$.

The maximum mass of a star is set by the requirement that the star must not destroy itself!

We derive the upper limit on the mass of stars by finding the mass at which hydrostatic equilibrium is violated, this happens when the photon pressure on the gas exceeds the gravitational force. Equating these forces therefore gives the maximum luminosity before hydrostatic equilibrium is violated,

$$L_r = \frac{4\pi c G M_r}{\kappa} = L_{\mathrm{Edd}}, \tag{73}$$

the Eddington luminosity - the point at which radiation pressure equals gravitational force.

From Eq. 9, we can derive the upper limit to the mass of a main-sequence star,

$$\frac{M_{\mathrm{max}}}{M_\odot} = \left(\frac{4\pi c G M_\odot}{\kappa L_\odot}\right)^{\alpha-1}. \tag{74}$$

12 Pulsating Stars

Pulsating stars are classified as either predominantly propagating through the star (radial), or propagating around the star (non-radial). We focus on the radial mode.

12.1 Cepheid Variables

There is a strong correlation between period and luminosity for Cepheid variables,

$$\log_{10}\left(\frac{\langle L\rangle}{L_\odot}\right) = 1.15\log_{10}\Pi^d + 2.47. \tag{75}$$

This tight relationship allows us to use Cepheid variables as 'distance candles', from which we can calculate the distance to their galaxies from period and flux measurements.

12.2 Physics of Stellar Pulsation

The radial oscillations of a pulsating star are the result of sound waves resonating in the star's interior. Considering how long it would take a sound wave to cross the diameter of a star of radius R and constant density ρ, we find

$$\Pi \approx \sqrt{\frac{3\pi}{2\gamma G\rho}}. \tag{76}$$

Not all stars pulsate, this is explained by the valve mechanism. An opaque layer is required to 'dam up' energy flowing towards the surface, and push the layers upwards. As the expanding layer becomes transparent, the trapped heat will escape and the layer will fall back to begin the next cycle.

For this to work, opacity must increase with compression, meaning opacity must increase as both ρ and T increase. Opacity increases with temperature from excitation (bound-bound), to ionisation (bound-free). Partial ionisation zones are regions where only a fraction of the gas is ionised, and hence the opacity is high.

Part of the work done during the compression produces further ionisation rather than significantly raising the temperature, then during expansion the temperature decreases slowly as ions recombine and release energy (causing luminosity to increase).

All stars have partial ionisation zones, however they can't be too close to the surface as there is too little material to drive pulsations, nor too deep and convection occurs in the outer regions, damping out the pulsations.

12.3 The Instability Strip

Pulsating stars are inherently unstable, and the areas of the HR diagram on which they are found is known as the instability strip.

Pulsation period decreases as you move down the instability strip from the very tenuous supergiants to the very dense white dwarfs, this is explained by the period equation above.

13 Stellar Formation

13.1 Interstellar Medium

The material in galaxies not within stars or planets, it is a mixture of gas of different phases, and dust (silica and carbon particles).

There are three phases,

- $T < 300$K: Cold dense phase - comprised of neutral and molecular hydrogen
- $1000 < T < 10000$K: Warm phase - neutral and ionised gas
- $T > 1000000$K: Dynamic hot phase, gas shock-heated by supernovae

13.2 Cloud Collapse

Giant molecular clouds are a vast assemblage of molecular gas that has mass $M > 10,000 M_\odot$, they are concentrated in the planes of galaxies.

Stellar formation occurs when a mass of gas exceeds a critical density and begins to collapse. While collapsing, the cloud will fragment into smaller stars until protostars are formed. For stars to form, gravitational pressure must overcome thermal pressure, often a strong external pressure, such as shocks from supernovae, is required to initiate the collapse.

Assuming a spherical cloud of constant density, and on the basis of the Virial Theorem (see lecture 4), the condition for collapse $2K < |U|$ is

$$\frac{3M_c kT}{\mu m_H} < \frac{3GM_c^2}{R_c}. \tag{77}$$

Rewriting R_c in terms of cloud density and mass, we can solve for the minimum mass required for spontaneous collapse - the **Jeans mass**,

$$M_J \approx \left(\frac{5kT}{G\mu m_H}\right)^{\frac{3}{2}} \left(\frac{3}{4\pi\rho_0}\right)^{\frac{1}{2}}. \tag{78}$$

The Jeans mass is the *minimum mass for instability*. Lower M_J means collapse more probable, therefore gas cloud most likely to collapse when cold and dense. At high temperatures, there is often too much kinetic energy for the cloud to collapse (unless very massive).

The initial phase of the gravitational collapse is likely to be in free fall, with timescale

$$t_{\text{ff}} = \left(\frac{3\pi}{32G\rho_0}\right)^{\frac{1}{2}}, \tag{79}$$

the cloud will be optically thin, therefore radiation from the collapse can escape and temperature does not increase. When the cloud becomes optically thick to radiation, the temperature increases and the collapse slows due to an increase in internal pressure (which was negligible before). A point is reached where the internal pressure prevents further collapse, and this provides a lower limit to the masses of the cloud fragments ($\approx 0.1 M_{\odot}$).

13.3 Protostars

As the cloud collapses in free fall, it will fragment as individual regions exceed the local Jeans mass. The increasing density of the collapsing cloud fragments eventually makes the gas opaque to infrared photons, radiation is trapped and the centre of the cloud begins to heat. As the centre heats, the pressure increases and eventually the fragments reach hydrostatic equilibrium - making them *protostars*.

13.4 Accretion

Surrounding gas keeps falling onto the protostellar core, conservation of angular momentum causes this gas to form an accretion disk around the protostar. This process generates gravitational energy, part of which further heats the core and part of which is radiated away, contributing to luminosity.

13.5 Dissociation and Ionisation

Initially, the gas is composed of molecular Hydrogen, as it is compressed temperatures rise and hydrogen is ionised, causing the gas cloud to collapse faster. At some point, hydrogen is completely ionised and the cloud behaves like an ideal gas again, hydrostatic equilibrium is restored. Note once this process is complete radius is much lower.

13.6 Pre-main sequence

Accretion slows down, eventually stopping, and the protostar is revealed as a pre-main sequence star, with its luminosity now provided by gravitational contraction. Further evolution takes place on the thermal timescale, the core heats as per the Virial theorem.

13.7 Hayashi Track

Curves on the HR diagram, these are the tracks protostars follow until the core temperature becomes high enough for nuclear fusion to occur and they join the main sequence.

14 Stellar Evolution: Low-Intermediate Mass Stars

The main factors that drive the evolution of stars are

- Abundance changes - temperature and density increase in the core
- Fusion size increase - luminosity increased
- Slow gravitational contraction
- Fast gravitational collapse when fusion ceases

The lifetime that a star undergoes nuclear fusion for a given fusion process will be

$$T = \frac{X\xi Mc^2}{L} \tag{80}$$

where X is the fraction of mass in the star that will be used in the fusion process, ξ is the mass-to-light efficiency.

From this we can solve for the main-sequence lifetime of stars, using $L \propto M^{\alpha}$, $\alpha \approx 3.5$,

$$T = 10^{10} \left(\frac{M_{\odot}}{M} \right)^{\alpha - 1} \text{years.} \tag{81}$$

14.1 Brown and Red Dwarfs

Brown dwarfs have masses $M < 0.08M_{\odot}$ and radiate through gravitational collapse - essentially protostars that never achieve nuclear fusion.

Red dwarfs are stars of $0.08 < M < 0.4m_{\odot}$. They undergo Hydrogen fusion but never achieve the core temperatures required for further fusion. Eventually become white dwarfs.

14.2 Evolution of low-intermediate mass stars

There is a balance between gravitational and thermal pressure (nuclear fusion) with a change in chemical composition with time.

Stars are characterised on the HR diagram by

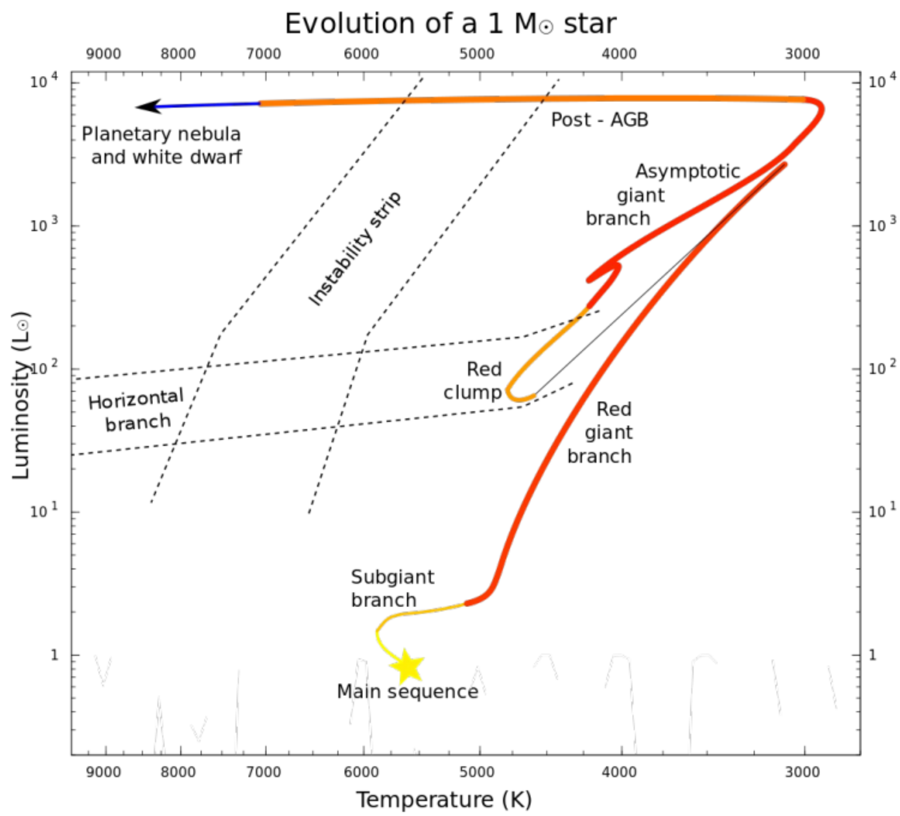
$$L = 4\pi R^2 \sigma T_e^4 \tag{82}$$

We see

- Shell burning: during H-H phase star contracts slowly due to chemical changes, outer regions of the core will heat up and Hydrogen fusion will commence in a shell around the core. Star increases in size and luminosity increases, radius increases and thus temperature will drop.

- Helium core burning: in degenerate conditions Helium ignites, temperature increases (Helium flash)
- Helium fusion: after H fusion (interval where star heats up enough to fuse He via triple alpha process). Lasts for less time as Helium fusion 10x less efficient. Star becomes very luminous
- Helium shell burning: (thermally pulsing AGB phase) e.g., Mira variables
- Planetary nebula: The outer envelopes of the star are expelled, this gas is excited and ionised by the photons from the inner white dwarf.

General pattern is **CORE FUSION** → **SHELL FUSION** → **CONTRACTION** and repeat.



15 Stellar Evolution: Massive Stars and Supernovae

Higher mass stars are able to fuse heavier elements (beyond Carbon and Oxygen). Their luminosities do not evolve as much as lower mass stars since they are already close to their Eddington limit.

Massive stars on the MS are bluer and will have radii 5-35x larger than the Sun, they have shorter lifetimes than their less massive counterparts.

15.1 Mass Loss

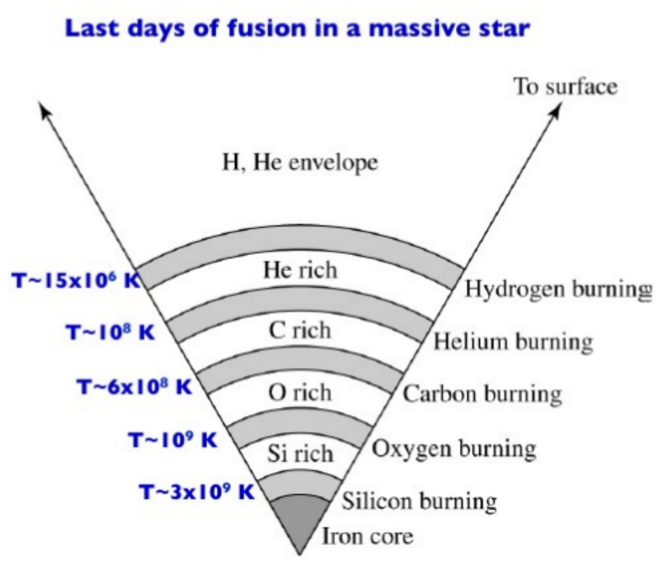
Massive stars suffer significant mass loss, particularly during the later nuclear fusion phases.

Above a certain mass ($M > 15M_\odot$), stellar winds affect the evolution.

Two mechanisms, Radiation driven and Red supergiant mass loss.

15.2 Nuclear Fusion

We see fusion shells of different elements.



Fusion of each subsequent element requires greater temperature and lasts for less time.

15.3 Advanced Evolution

When $T > 5 \times 10^8$K, neutrino losses are the most important energy leak.

Photo-disintegration occurs when thermal photons produced in the plasma in the core have so much energy (due to such high temperatures) that they can disintegrate the heavy elements, 'undoing' the work of previous nuclear fusion phases. Photodisintegration phases are endothermic, more energy goes in than out, so critically weaken the core - removing much of the energy that was holding up the rest of the star.

Electron capture occurs when free electrons are captured by the photo-disintegration products, rendering them unable to support the collapsing star via electron degeneracy pressure. This requires extremely high density, and leads to a colossal release of energy in neutrinos,

$$p^+ + e^- \rightarrow n + \nu_e \tag{83}$$

this process ultimately leads to a degenerate remnant, a neutron star or black hole as the core is now unable to support the star and collapses astonishingly quickly at *approximately* the free-fall timescale.

15.4 Rebound of the Stellar Core

The core collapses until it reaches a density of 8×10^{17}kgm^{-3}. At this point, there is a repulsion due to the nuclear strong force, causing the core to rebound with tremendous energy, sending out a shock wave.

Shock wave loses energy in the outer layers of the star, further photo-disintegration and electron capture occurs producing even more neutrinos, gas density is so high these neutrinos are deposited into the material, accelerating the gas and releasing energy in a **supernova**!

Elements heavier than iron are fused in this shock wave-gas interaction.

Gravitational energy released in the collapse given by

$$E \approx \frac{3}{10}\frac{GM^2}{R} \tag{84}$$

when $R \ll R_{\text{initial}}$.

15.5 Remnants

Collapse core becomes neutron star or black hole dependant on mass. We see large nebulae around the dead star.

16 Stellar Evolution: Endpoints

16.1 Basic Properties of White Dwarfs

Low luminosity, high temperature, extremely high core pressure. Mass radius relationship

$$R \propto M^{-\frac{2}{3}} \tag{85}$$

implying higher mass white dwarfs have lower radii. To withstand increased gravitational force, the momentum of the electrons must increase to increase degeneracy pressure.

The **Chandrasekhar limit** is the mass limit (of the stellar core) where electron degeneracy fails, $1.4M_{\odot}$, and the core collapses.

16.2 Cooling in White Dwarfs

White dwarfs cool due to the release of kinetic energy from their nuclei, on the order of a billion years. As white dwarfs cool they follow a distinct line on the HR diagram since their radius remains fixed (depends on mass). Ultimately a white dwarf becomes sufficiently cool that it doesn't produce significant optical emission, a **black dwarf**.

16.3 Basic Properties of Neutron Stars

Stellar cores more massive than the Chandrasekhar limit but less than $2.5M_{\odot}$ collapse to form neutron stars.

Held together by neutron degeneracy pressure and repulsion from the strong force. Can be considered a very large atomic nucleus. Has the mass of a typical star but much, much smaller radius.

Surface temperature 10^6K.

17 Pulsars and Black Holes

17.1 Pulsars

Pulsars are believed to be rotating neutron stars due to the collapse of the stellar core of massive stars. They have well defined pulse periods of between 0.25 and 2 seconds, very short implying the rotations are extremely fast. The period of pulsars increase very gradually as the pulses slow down.

Maximum angular velocity can be found by equating the gravitational and centripetal accelerations at the equator, where the constraint on angular velocity is most important as stellar material moves most rapidly (star must be kept from flying apart). Assuming uniform density ρ, we find the minimum period is given by

$$P_{\min} = \left(\frac{3\pi}{G\rho}\right)^{\frac{1}{2}}. \tag{86}$$

Note, the limit on angular velocity is actually higher as the star will distort into an ellipsoid and tend to lose material in a disk-like extension of the equatorial region.

This relation shows that the density required for such short pulse periods is that of a neutron star.

Charged particles in neutron star's magnetosphere are accelerated to relativistic speeds along the field lines, emitting radiation in a narrow beam.

17.2 Black Holes

When neutron degeneracy and strong force fails, no known force can prevent the object from total collapse, the object becomes a black hole.

The Schwarzschild radius is the distance from the centre of the black hole where the escape velocity is the speed of light.

$$R_c = \frac{2GM}{c^2}. \tag{87}$$

Black holes emit Hawking radiation until they completely 'evaporate', but this takes an extremely long time.